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Extension of Beta Function and its Agricultural Applications

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ABSTRACT

The first kind of Euler integral, known as the β-function, holds significant importance in various fields such as mathematics, engineering, statistics, and the chemical and physical sciences. Numerous extensions of the classical β-function can be found in the literature. In the realm of mathematics, the β-function plays a crucial role in addressing problems related to probability distribution, quantum mechanics, and fluid mechanics. This study aims to introduce new extensions of the β-function with a specific focus on their applications in agriculture. Additionally, the research delves into the applications of the β-function and explores various types of β-functions in conjunction with other special functions. The study further investigates additional extensions, properties, and mutual relationships, shedding light on recurrence relations and generating functions associated with the extended β-function.

Keywords: Euler Beta-function, Mathematical Applications, Agricultural Extensions, Special Functions, Recurrence Relations.

INTRODUCTION

New extension of extented $\beta -$ function is most important form of special function, due to the used in various flied like physics, engineering, statistical science and mathematics.Recently many work of β function,moreover researches more interest to derived the application of (β-function,Gamma function).

[Rainville, 1960] has defined the Euler β-function.

$$B(\xi,\eta) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1}dx\,, (R(\xi) > 0, R(\eta) > 0) \quad (1)$$

[Chaudhry and Zubair, 2002] has presented the *Gamma*-function. The Γ-function analyzed the β Function. Also, explained the relationship between β and Gamma function. Gamma functions are obtained as

$$\Gamma(\xi) = \int_0^\infty x^{\xi-1}exp(-x)dx\,, (Re(\xi) > 0) \quad (2)$$

and

$$B(\xi,\eta) = \frac{\Gamma(\xi)\Gamma(\eta)}{\Gamma(\xi+\eta)}\,, (R(\xi) > 0, R(\eta) > 0). \quad (3)$$

where $(\gamma)_k$are the pochhammer symbol to have [Srivastava, 2012].

$$(\zeta)_k = \begin{cases} 1 \quad (k=0) \\ \zeta(\zeta+1)\ldots(\zeta+k-1)\ (k\epsilon N) \end{cases} \quad (4)$$

[Chaudhry et~al., 1997] has presented the extension of extended β-Function .

$$B(\xi,\eta;z) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1}exp\left(-\frac{z}{x(1-x)}\right)dx, \quad (5)$$

$$Re(\xi) > 0\,, R(\eta) > 0, z \geq 0.$$

[Choi et~al., 2014] has introduced the extension of extended β function are explained by

$$B(\xi,\eta;u,z) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1}exp\left(\frac{-z}{x} - \frac{u}{(1-x)}\right)dx, \quad (6)$$

$$Re(\xi) > 0\,, R(\eta) > 0, z, u \geq 0$$

[Khan and Ghayasuddin, 2016] has established the extension of β function is explaned by

$$B(\xi,\eta;u,z)=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}exp\left(\frac{-z}{x^m}-\frac{u}{(1-x)^m}\right)dx, \quad (7)$$

$Re(\xi)>0\,, R(\eta)>0, m,z,u\geq 0$

[Atash and Barahmah, 2017] has derived the extension of β and gamma function is defined by

$$B(\xi,\eta;u,t)=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}E_{\alpha,\beta}\left(\frac{-z}{x}\right)E_{\alpha,\beta}\left(-\frac{u}{(1-x)}\right)dx, \quad (8)$$

$Re(\xi)>0\,, R(\eta)>0, Re(z)>0, Re(u)>0, \alpha,\beta>0$

$$\Gamma(\xi)=\int_0^\infty x^{\xi-1}E_{\alpha,\beta}(-x)dx, Re(\xi)>0, \alpha,\beta>0 \quad (9)$$

In this research, the introduced the wright function $W_{\alpha,\beta}$ [Kilbas et~al., 2006] is defined as

$$W_{\alpha,\beta}(t)={}_1\psi_1=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{t^k}{k!} \quad (10)$$

$\alpha,\beta\varepsilon C$ ` $and Re(\alpha)>0, Re(\beta)>1$

[Ata, 2018] has established the extension of gamma and β function is defined as

$$^{\psi}\Gamma_{z,u}^{\alpha,\beta}(\xi)=\int_0^\infty x^{\xi-1}\,{}_1\psi_1\left(\alpha,\beta;-x-\frac{z}{x}\right)dx. \quad (11)$$

$,Re(\xi)>0, Re(\alpha)>0, Re(\beta)>1, Re(z)>0$

and

$$^{\psi}B_{Z,u}^{\alpha,\beta}(\xi,\eta)=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}\,{}_1\psi_1\left(\alpha,\beta;-\frac{z}{x(1-x)}\right)dx\,, \quad (12)$$

$(R(\xi)>0, R(\eta)>0, Re(\alpha)>0, Re(\beta)>1, Re(z)>0$

In this research article, we introduced the new extension of extended β function and gamma function.

$$^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta)=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx\,, \quad (13)$$

$(R(\xi)>0, R(\eta)>0, Re(\alpha)>0, Re(\beta)>1, Re(z)>0, Re(u)>0, m>0$

and

$$^{\omega}\Gamma_{z,u}^{\alpha,\beta,m}(\xi)=\int_0^\infty x^{\xi-1}\,\omega_{\alpha,\beta}\left(-x^m-\frac{z}{x^m}\right)dx. \quad (14)$$

$,Re(\xi)>0, Re(\alpha)>0, Re(\beta)>1, Re(z)>0, m>0$

we define the new extension of extended β and gamma function as ω-Beta and ω-gamma function.If $m=1$ then

$$^{\omega}\Gamma_{z,u}^{(0,1)}(\xi)={}^{\omega}\,\Gamma_{z,u}(\xi)$$

$$^{\omega}\Gamma_{(0,0)}^{(0,1)}(\xi)={}^{\omega}\,\Gamma(\xi)$$

and

$$^{\omega}B_{Z,u}^{(0,1)}\xi,\eta)={}^{\omega}\,B_{Z,u}(\xi,\eta)$$

$$^{\omega}B_{(0,0)}^{(0,1)}(\xi,\eta)={}^{\omega}\,B^{(0,1)}(\xi,\eta)$$

Remark 1

- if $\alpha=\beta=1$ and $m=1$ then (13) reduced into extended β function(6).
- if $z=u$, and $m=1$ then (13) reduced into extendedβ function(5).
- if $\alpha=\beta=1$,$z=u=0$ and$m=1$ then (13) reduced into classical β function(1).

Theorem 1; *The new extension of extended β function to derived the form of relation from mellin transformation:*

$$M\{^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta);z\longrightarrow s,u\longrightarrow r\}=B(\xi+ms,\eta+mr)\Gamma^{\alpha,\beta,m}(s)\Gamma^{\alpha,\beta,m}(r) \quad (15)$$

$Re(\xi+ms)>0$,$Re(\eta+mr)>0$,$Re(s)>0$,$Re(r)>0$,$Re(z)>0$,$Re(u)>0$,

$Re(\alpha)>0$,$Re(\beta)>1$.

proof: By using Equation (13), applying both side mellin transformation.

$$M\{^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta);z\longrightarrow s,u\longrightarrow r\}=\int_0^\infty\int_0^\infty z^{s-1}u^{r-1}\times$$

$$\left(\int_0^1 x^{\xi-1}(1-x)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx\right)dzdu \quad (16)$$

$$M\{{}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta); z\longrightarrow s, u\longrightarrow r\} = \int_0^1 x^{\xi-1}(1-x)^{\eta-1}\times$$

$$\{\int_0^{\infty} z^{s-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)dz\}\{\int_0^{\infty} u^{r-1}\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)\} \quad (17)$$

Substituting in equation (17) $p=\frac{z}{x^m}$ and $q=\frac{u}{(1-x)^m}$. we have that

$$M\{{}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta); z\longrightarrow s, u\longrightarrow r\} = \int_0^1 x^{\xi+ms-1}(1-x)^{\eta+mr-1}\times$$

$$\{\int_0^{\infty} z^{s-1}\omega_{\alpha,\beta}(-p)dz\}\{\int_0^{\infty} u^{r-1}\omega_{\alpha,\beta}(-q)\} \quad (18)$$

Equation(18) Applying the Dafination of gamma.we obtain that

$M\{{}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta); z\longrightarrow s, u\longrightarrow r\} = B(\xi+ms,\eta+mr)\Gamma^{\alpha,\beta,m}(s)\Gamma^{\alpha,\beta,m}(r)$.

2 Corollary

The holds the integral representation are true;

$$\int_0^{\infty}\int_0^{\infty} {}^{B}{}_{z,u}(\xi,\eta)dzdu = B(\xi+1,\eta+1) \quad (19)$$

Proof: We know that

$$M\{{}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta); z\longrightarrow s, u\longrightarrow r\} = \int_0^{\infty}\int_0^{\infty} B_{z,u}^{\alpha,\beta,m}(\xi,\eta)dzdu$$

$$= B(\xi+ms,\eta+mr)\Gamma^{\alpha,\beta,m}(s)\Gamma^{\alpha,\beta,m}(r) \quad (20)$$

$Re(\xi+ms)>0, Re(\eta+mr)>0, Re(s)>0, Re(r)>0, Re(z)>0, Re(u)>0,$
$Re(\alpha)>0, Re(\beta)>1$

By substituting in equation (16), $r=s=1$ and $\alpha=\beta=1, m=1$, and $\Gamma^{\alpha,\beta}(1)=\frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}$ and also, $\Gamma(1)=1$ we are obtained the result

$$\int_0^{\infty}\int_0^{\infty} B_{z,u}(\xi,\eta)dzdu = B(\xi+1,\eta+1)$$

Theorem 2: *For the new extension of extended βfunction,we have derived the integral representations:*

$${}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = 2\int_0^{\frac{\pi}{2}}\cos^{2\xi-1}\theta\sin^{2\eta-1}\theta\,\omega_{\alpha,\beta}\left(-\frac{z}{\cos^{2m}\theta}\right)\omega_{\alpha,\beta}\left(-\frac{u}{\sin^{2m}\theta}\right)d\theta.$$

$${}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = \int_0^{\infty}\frac{v^{\xi-1}}{(1+v)^{\xi+\eta}}\omega_{\alpha,\beta}\left(-\frac{z(1+v)^{2m}}{v^m}\right)\omega_{\alpha,\beta}(-u(1+v)^m)dv$$

$${}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = 2^{1-\xi-\eta}\int_{-1}^{1}(1-v)^{\xi-1}(v)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{2^m z}{(1+v)^m}\right)\omega_{\alpha,\beta}\left(-\frac{2^m u}{(1-v)^m}\right)dv.$$

and

$${}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = (c-a)^{1-\xi-\eta}\int_a^c (v-a)^{\xi-1}(c-v)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z(c-a)^m}{(v-a)^m}\right)\omega_{\alpha,\beta}\left(-\frac{u(c-a)^m}{(c-v)^m}\right)dv,$$

$z,u,m\geq 0, \alpha,\beta>0, Re(z)>0, Re(u)>0, Re(\alpha>0), Re(\beta)>1, Re(\xi)>0, Re(\eta)>0.$

Proof: Taking substitution in Equation (13) $x=\cos^2\theta$, variable are changed then changed the limit $0\longrightarrow 1$ from $0\longrightarrow\frac{\pi}{2}$. so we obtained as

$x=\cos^2\theta$
$dx=2\cos\theta\sin\theta d\theta$

$${}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = 2\int_0^{\frac{\pi}{2}}\cos^{2\xi-1}\theta\sin^{2\eta-1}\theta\,\omega_{\alpha,\beta}\left(-\frac{z}{\cos^{2m}\theta}\right)\omega_{\alpha,\beta}\left(-\frac{u}{\sin^{2m}\theta}\right)d\theta.$$

Taking substitution in Equation (13) $x=\frac{v}{1+v}$ and $dx=(1+v)^{-2}$ they variable are changed then changed the limit $0\longrightarrow 1$ from $0\longrightarrow\infty$. so we get as

$${}^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = \int_0^{\infty}\frac{v^{\xi-1}}{(1+v)^{\xi+\eta}}\omega_{\alpha,\beta}\left(-\frac{z(1+v)^{2m}}{v^m}\right)\omega_{\alpha,\beta}(-u(1+v)^m)dv$$

Taking substitution in Equation (13) $x=\frac{1+v}{2}$, there variable are changed then changed the limit $0\longrightarrow 1$ from $1\longrightarrow -1$. so we have that

$$^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta)=2^{1-\xi-\eta}\int_{-1}^{1}(1+v)^{\xi-1}(1-v)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{2^m z}{(1+v)^m}\right)\omega_{\alpha,\beta}\left(-\frac{2^m u}{(1-v)^m}\right)dv.$$

Taking substitution in Equation (13) $x=\frac{v-a}{c-a}$, there variable are changed then changed the limit $0\longrightarrow 1$ from $c\longrightarrow a$. so we now that,

$$^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta)=(c-a)^{1-\xi-\eta}\int_{a}^{c}(v-a)^{\xi-1}(c-v)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z(c-a)^m}{(v-a)^m}\right)\omega_{\alpha,\beta}\left(-\frac{u(c-a)^m}{(c-v)^m}\right)dv,$$

Theorem 3 : *The functional relation of extended β function are holds;*

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,\eta)+{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta+1)={}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta)$$

proof :

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,\eta)+{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta+1)=\int_0^1 x^{\xi}(1-x)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$
$$+\int_0^1 x^{\xi-1}(1-x)^{\eta}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx\ ,$$

$$=\int_0^1 [x^{\xi}(1-x)^{\eta-1}+x^{\xi-1}(1-x)^{\eta}]\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$
$$=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx={}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta)$$

2 Corollary

These function are holds on β function.

$$B_{z,u}(\xi+\eta+1)+B_{z,u}(\xi+1,\eta)=B_{z,u}(\xi,\eta)$$

Proof : if putting $\alpha=\beta=1$ and $m=1$ in theorem (3). then we get result.

3 Corollary

These function are holds on β function.

$$B_z(\xi+\eta+1)+B_z(\xi+1,\eta)=B_z(\xi,\eta)$$

Proof: if putting $\alpha=\beta=1,z=u$ and $m=1$ in theorem (3). then we get result.

4 Corollary

These function are holds on β function.

$$B(\xi+\eta+1)+B(\xi+1,\eta)=B(\xi,\eta)$$

Proof : if putting $\alpha=\beta=1,z=u=0$ and $m=1$ in theorem (3). then we get result.

Theorem 4: *The second functional relation of extended β function are holds;*

Proof : $^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+2,\eta)+2\,{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,\eta+1)+{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta+2)={}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta)$

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+2,\eta)+2\,{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,\eta+1)+{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,\eta)=\int_0^1 x^{\xi+1}(1-x)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$
$$+2\int_0^1 x^{\xi}(1-x)^{\eta}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$
$$+\int_0^1 x^{\xi-1}(1-x)^{\eta+1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$
$$=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}[t^2+2t(1-t)+(1-t)^2]\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$
$$=\int_0^1 x^{\xi-1}(1-x)^{\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx\ ={}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta)$$

Theorem 5: *The summation formulas are holds on the extension of β function.*

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,1-\eta)=\sum_{k=0}^{\infty}\frac{(\eta)_k}{k!}{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+\nu,1)$$
$$Re(\alpha)>0,Re(\beta)>1,Re(z)>0,Re(u)>0,m>0$$

proof :

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,1-\eta)=\int_0^1 x^{\xi-1}(1-x)^{1-\eta-1}\omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right)\omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right)dx$$

By applying Binomial series.

$$(1-x)^{-\eta} = \sum_{v=0}^{\infty} (\eta)_v \frac{(x)^v}{v!}$$

then we have that;

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi, 1-\eta) = \int_0^1 \sum_{v=0}^{\infty} (\eta)_v \frac{1}{v!} x^{\xi+v-1} \omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right) \omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right) dx$$

Hence obtain the result

$$= \sum_{v=0}^{\infty} (\eta)_v \frac{1}{v!}\, {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+v, 1)$$

Theorem 6 : *The infinite summation formulas are satisfies on β function the are following;*

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta) = \sum_{k=0}^{\infty} {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+k, \eta+1)$$

Proof: $^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1} \omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right) \omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right) dx$

The series representation as;

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \quad if|t| < 1$$

By integration and summation order are interchange,and series are uniformly converges.

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta) = \sum_{k=0}^{\infty} \int_0^1 x^{\xi-1}(1-x)^{\eta+1-1} \omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right) \omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right) dx$$

Hence proved the result.

$$= \sum_{k=0}^{\infty} {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+k, \eta+1)$$

Theorem 7 : *The extended β function are satisfies the given relation;*

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta) = \sum_{n=0}^{m} \binom{m}{n} {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+n, \eta+m-n), (n\epsilon N)$$

proof The above relation,

$$^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = \int_0^1 x^{\xi-1}(1-x)^{\eta-1} \omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right) \omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right) dx$$

$$^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = \int_0^1 x^{\xi-1}(1-x)^{\eta+m-1}(1-x)^{-m} \omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right) \omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right) dx, \quad (21)$$

We known that;

$$(1-x)^{-m} = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n$$

By use in (21), and replaced $m \longrightarrow m-n$ so, we have that;

$$^{\omega}B_{Z,u}^{\alpha,\beta,m}(\xi,\eta) = \sum_{n=0}^{m} \binom{m}{n} \int_0^1 x^{\xi+n-1}(1-x)^{\eta+m-n-1} \omega_{\alpha,\beta}\left(-\frac{z}{x^m}\right) \omega_{\alpha,\beta}\left(-\frac{u}{(1-x)^m}\right) dx$$

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,\eta) = \sum_{n=0}^{m} \binom{m}{n} {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+n, \eta+m-n)$$

Theorem 8: *The β function are satisfies the relation;*

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi, -\xi-m) = \sum_{n=0}^{m} \binom{m}{n} {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+n, -\xi-n), (n\epsilon N)$$

Proof: The given relation,

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(s+1,t) + {}^{\omega}B_{z,u}^{\alpha,\beta,m}(s,t+1) = {}^{\omega}B_{z,u}^{\alpha,\beta,m}(s,t)$$

By Taking $m = 1,2,3,\ldots.,$

so that firstly substituting $m = 1$ by given relation.

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,-\xi-1) = {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,-\xi) + {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,-\xi-1)$$

Secondly substituting $m = 2$ by given relation.

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,-\xi-2) = {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,-\xi) + 2\,{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,-\xi-1) + {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+2,-\xi-2)$$

Thirdly substituting $m = 3$ by given relation.

$$^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,-\xi-3) = {}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi,-\xi) + 3\,{}^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+1,-\xi-1)$$

$$+3^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+2,-\xi-2)+^{\omega}B_{z,u}^{\alpha,\beta,m}(\xi+3,-\xi-3)$$

Exploring the extended β-function and its applications in agriculture is a big step towards bridging mathematical theory with practical issues. The classical β-function is a basic tool for tackling complicated issues across numerous scientific areas. This work expands on the β-function and applies it to agriculture, highlighting its adaptability and significance in solving real-world problems. The classical beta function plays an important role in various fields, including mathematical, physical, engineering, and statistical sciences (Chand, 2017).

In mathematics, the β-function is commonly used to represent and solve issues in probability distribution, quantum physics, and fluid mechanics. This work introduces additional expansions of the β-function, perhaps leading to larger applications in various fields. Examining the characteristics, mutual interactions, recurrence relations, and generating functions of the extended β-function enriches the mathematical landscape and provides useful insights for future study in numerous scientific areas. Al-Hussaini *et al*. (2005) provide recurrence relations for a moment and conditional moment generating functions of generalized order statistics, enabling the classification of distributions like Weibull, Pareto, power function, Gompertz, and compound Gompertz distributions.

This study's focus on agricultural applications differs from standard uses of the β-function. Agriculture, as a critical area for world sustainability, may greatly benefit from the use of modern mathematical principles. This research uses the β-function to address agricultural concerns, bridging the gap between theoretical mathematics and practical challenges encountered by the agricultural sector. The Î2-function can bridge the gap between theoretical mathematics and practical challenges in agriculture, promoting sustainable agriculture (Machwitz *et al*., 2021).

Experimenting with different β-functions and special functions reveals their versatility in tackling various problems. The combination of multiple mathematical functions allows for more detailed knowledge of agricultural processes, perhaps leading to novel solutions and greater farming efficiency. In a study Moreenthaler *et al*. (2003) reported that a constrained optimization algorithm can improve remote sensing/precision agriculture by maximizing farmer's profit while decreasing environmental damage and costly treatments.

The study's exploration into extending the β-function demonstrates the dynamic nature of mathematical research. As new extensions are proposed and investigated, the opportunities for practical applications grow, providing new views and approaches for tackling difficult challenges in agriculture and others. This is complemented by Gao *et al*. (2020) that the new agricultural technology extension mode improves technology adoption levels of farmers, with different ages and farmland sizes benefiting differently.

This paper makes a significant addition to mathematical theory and practical problem-solving by extending the β-function and applying it to agriculture. This discovery broadens the application of the β-function, improving our grasp of mathematical functions and providing essential tools for tackling agricultural difficulties. As multidisciplinary research evolves, collaboration between mathematics and agriculture shows potential for creating long-term solutions and advances in farming techniques.

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